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# Camassa–Holm, Korteweg–de Vries and related models for water waves

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In this paper we first describe the current method for obtaining the Camassa–Holm equation in the context of water waves; this requires a detour via the Green–Naghdi model equations, although the important connection with classical (Korteweg–de Vries) results is included. The assumptions underlying this derivation are described and their roles analysed. (The critical assumptions are, (i) the simplified structure through the depth of the water leading to the Green–Naghdi equations, and, (ii) the choice of submanifold in the Hamiltonian representation of the Green–Naghdi equations. The first of these turns out to be unimportant because the Green–Naghdi equations can be obtained directly from the full equations, if quantities averaged over the depth are considered. However, starting from the Green–Naghdi equations precludes, from the outset, any role for the variation of the flow properties with depth; we shall show that this variation is significant. The second assumption is inconsistent with the governing equations.)

Returning to the full equations for the water-wave problem, we retain both parameters (amplitude, $\varepsilon$, and shallowness, $\delta$) and then seek a solution as an asymptotic expansion valid for, $\varepsilon \to 0$, $\delta \to 0$, independently. Retaining terms $O(\varepsilon)$, $O(\delta^2)$ and $O(\varepsilon\delta^2)$, the resulting equation for the horizontal velocity component, evaluated at a specific depth, is a Camassa–Holm equation. Some properties of this equation, and how these relate to the surface wave, are described; the role of this special depth is discussed. The validity of the equation is also addressed; it is shown that the Camassa–Holm equation may not be uniformly valid: on suitably short length scales (measured by $\delta$) other terms become important (resulting in a higher-order Korteweg–de Vries equation, for example). Finally, we indicate how our derivation can be extended to other scenarios; in particular, as an example, we produce a two-dimensional Camassa–Holm equation for water waves.

## 1. Introduction

From the earliest days in the development of what we now commonly refer to as soliton theory, many competing soliton-type models for water waves have been suggested, a few of which have origins that pre-date the solution of the Korteweg–de Vries (KdV) equation. These models have attempted to capture one aspect or another of the classical water-wave problem, and not all the resulting equations are susceptible to soliton techniques. Nevertheless, this has often been the driving force behind these various investigations in recent years. Thus, even in the context of the simplest physical model for water waves, there are many variants of the Korteweg–de

Vries (KdV) equation:

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

for example, in two Cartesian dimensions, in cylindrical geometry or for head-on collisions. (A description of these KdV-type equations can be found in Johnson 1997; we present the various equations, at this stage, in a normalized form, merely as exemplars of what can be obtained; $x$ and $t$ have the usual interpretations.)

In addition, there are other models that do not follow the KdV route. So, for example, we have the shallow-water equations,

$$u_t + uu_x + h_x = 0, \quad h_t + (hu)_x = 0,$$

where $u(x,t)$ is the horizontal velocity component and $h(x,t)$ the total depth of the water. (This pair of equations possesses an infinity of conservation laws; see Benny 1974; Miura 1974.) On the other hand, there are model equations which appear to follow the KdV pattern, but turn out to possess fundamentally different properties. Such an example is the BBM equation (Benjamin, Bona & Mahoney 1972, and first used in Peregrine 1966), sometimes called the regularized long-wave equation,

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{2}$$

More recently, an equation with an involved pedigree, but intriguing solutions, has appeared on the scene: the Camassa–Holm equation. This equation, in the same vein as above, is typically written as

$$u_t + 2\kappa u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}, \tag{3}$$

where $\kappa$ is a parameter; most consideration has been given to the case $\kappa = 0$ for which peaked solitons ('peakons') exist, which have excited some interest in soliton theory. Indeed, this equation was first introduced by Fokas & Fuchssteiner (1981) as part of a study of some general aspects of this theory. Equation (3), for all $\kappa$, is an integrable equation; see Camassa & Holm (1993) and Constantin (2001) (and also Cooper & Shepard (1994), where travelling-wave solutions for various $\kappa$ are described). This equation incorporates nonlinear dispersive terms (the right-hand side of the equation) in addition to those terms associated with the BBM equation, (2). Other equations, which bear some similarity to equation (3), also exist, but they are not relevant to the route we follow here; see, for example, Fornberg & Whitham (1978) and Rosenau & Hyman (1992). Much of the conventional background that provides a setting for the Camassa–Holm equation, and its connection with other similar nonlinear equations, is well-documented in Camassa, Holm & Hyman (1994).

We have alluded to the background of the Camassa–Holm (CH) equation. In the context of water waves, there appears to be no satisfactory basis for regarding this equation as a valid approximation obtained from a systematic (asymptotic) procedure applied to the full governing equations. Although a brief description of a procedure which appears to mirror our approach is given in Fokas (1995), this analysis seems to suppress the dependence on the vertical coordinate ($z$); we shall demonstrate that incorporating the correct $z$-dependence is critical in providing a complete and accurate account of this problem.

In this paper, we will first describe the current procedures for deriving the CH equation and, where useful, relate this to the – now classical – results that obtain for the KdV equation. This will necessitate an excursion in the direction of the Green–Naghdi equations (Green & Naghdi 1976), in one spatial dimension and as specific for our problem, together with a particular submanifold selection within a Hamiltonian description. This approach will enable us to make clear the various shortcomings of these models, and will emphasize the critical assumptions that must be made in order to proceed. In passing, we will also present the corresponding higher-order KdV results that are, in a sense, an analogue of the CH equation. We will then show that the Camassa–Holm equation does indeed arise in the water-wave problem, but under a careful limiting process and by working with $u(x,t,z)$ the horizontal velocity component, for a particular $z$.

Some properties of this equation, and how it relates to the description of the surface wave, will be discussed. It will also be suggested that the equation cannot always be expected to constitute a uniformly valid approximation even on the appropriate time and spatial scales where the balance occurs. This is because certain types of initial data are possible which indicate that higher-order spatial derivatives may be required. This might lead to the appearance of an appropriate higher-order KdV equation, as will be described. Finally, the possibility of extending the calculation to different scenarios is addressed; as an example, a two-dimensional Camassa–Holm equation (for water waves) is derived.

## 2. Governing equations

Our vehicle for describing a problem for which the CH equation is proposed as a model is wave propagation on the surface of water. The simplest scenario is that modelled by an inviscid fluid of constant depth, which is stationary in its undisturbed state; the effects of surface tension are also ignored. We take the governing equation to be Euler's equation (in two dimensions) with appropriate surface and bottom boundary conditions. This problem is non-dimensionalized using the undisturbed depth of water, $h$, as the vertical length scale, a typical wavelength of the wave, $\lambda$, as the horizontal length scale, and a typical amplitude of the wave, $a$; see figure 1. It then follows that the appropriate non-dimensionalization for the horizontal velocity component is $\sqrt{gh}$ (where $g$ is the acceleration due to gravity), with a corresponding time, $\lambda/\sqrt{gh}$. The resulting non-dimensional equations contain two parameters: $\varepsilon = a/h$, the amplitude parameter, and $\delta = h/\lambda$, the shallowness parameter. When we write the surface as $z = 1 + \varepsilon\eta(x,t;\varepsilon,\delta)$, and let $p$ be the (non-dimensional) pressure relative to the hydrostatic pressure in the undisturbed state, we have the equations

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \tag{4}$$

$$\delta^2\{w_t + \varepsilon(uw_x + ww_z)\} = -p_z, \tag{5}$$

$$u_x + w_z = 0, \tag{6}$$

with

$$p = \eta, \quad w = \eta_t + \varepsilon u\eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta, \tag{7}$$

and

$$w = 0 \quad \text{on} \quad z = 0. \tag{8}$$

(Subscripts denote partial derivatives throughout.)

We comment, for future reference, that the parameter $\delta$ can be removed from these equations, as is well known. That is, when we transform according to

$$x = \frac{\delta}{\sqrt{\varepsilon}}\chi, \quad t = \frac{\delta}{\sqrt{\varepsilon}}\theta, \quad w = \frac{\sqrt{\varepsilon}}{\delta}\hat{w}, \tag{9}$$

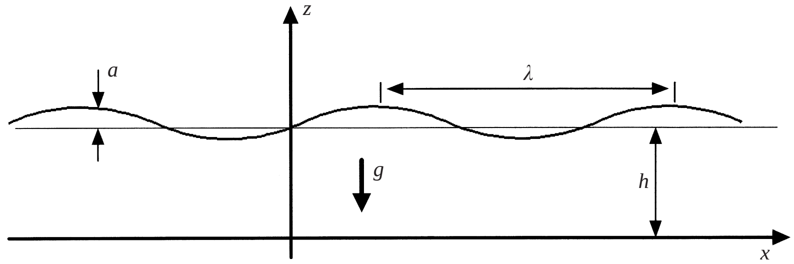


FIGURE 1. Defining sketch for the variables and scales used in the water-wave problem.

the equations (4)–(8) are recovered, but with $\delta^2$ replaced by $\varepsilon$ in equation (5). Then, for arbitrary $\delta$, we have the region of $(x, t)$-space where, for example, the linear, non-dispersive wave dominates as $\varepsilon \to 0$, i.e. where $\chi = O(1)$, $\theta = O(1)$. (This transformation, (9), is equivalent to using $h$ alone as the length scale in the problem.) Of course, the role of $\delta$ (independent of $\varepsilon$) in the original equations is still useful and important in the description of, for example, non-dispersive, but arbitrary-amplitude waves, i.e. $\delta \to 0$, $\varepsilon$ fixed. We shall see that each choice makes an important contribution in the story that unfolds. Finally, a familiar interpretation, which reinforces the importance of the relative roles of $\varepsilon$ and $\delta$, is that of initial data which defines a suitably small $\varepsilon$ and which then steepens as it evolves. When the local size of $\varepsilon$ (measuring the slope of the front) is about the size of $\delta^2$, then a balance ensues between nonlinearity and dispersion; this balance is essentially that given by the far-field description presented below (via the variables (10)).

First, before we explore the various new avenues in this problem, we comment briefly on the familiar and classical results that follow directly from equations (4)–(8). Consider the equations with $\delta^2$ replaced by $\varepsilon$ (by using (9)), then the leading-order problem, as $\varepsilon \to 0$, is simply

$$u_\theta = -p_\chi, \quad p_z = 0, \quad u_\chi + \hat{w}_z = 0,$$

with

$$p = \eta, \quad \hat{w} = \eta_\theta \quad \text{on} \quad z = 1$$

and

$$\hat{w} = 0 \quad \text{on} \quad z = 0.$$

These give, directly,

$$p = u = \eta, \quad \hat{w} = -z\eta_\chi \quad (0 \leqslant z \leqslant 1)$$

with

$$\eta_{\theta\theta} - \eta_{\chi\chi} = 0,$$

(where we have assumed that $u = 0$ wherever $\eta = 0$); this solution describes the linear, non-dispersive surface wave. If we now follow the right-going wave, say, and examine the problem in an appropriate far field, i.e. introduce

$$\xi = \chi - \theta, \quad \tau = \varepsilon\theta, \tag{10}$$

into the full equations, (4)–(8), then the leading-order surface wave satisfies (for $\xi = O(1)$, $\tau = O(1)$) the KdV equation

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \tfrac{1}{3}\eta_{0\xi\xi\xi} = 0, \tag{11}$$

where $\eta(\xi, \tau, \varepsilon) \sim \eta_0 + \varepsilon\eta_1$, and

$$p \sim \eta_0 + \varepsilon\{\eta_1 + \tfrac{1}{2}(1 - z^2)\eta_{0\xi\xi}\}, \tag{12}$$

$$u \sim \eta_0 + \varepsilon\{\eta_1 - \tfrac{1}{4}\eta_0^2 + (\tfrac{1}{3} - \tfrac{1}{2}z^2)\eta_{0\xi\xi}\}, \tag{13}$$

as $\varepsilon \to 0$. This result demonstrates that the balance required for the existence of the KdV equation will always arise in some region of physical space, subject only to the condition $\varepsilon \to 0$. This is to be compared with the derivations based on the special assumption $\delta^2 = O(\varepsilon)$, which would suggest that the KdV balance will occur only rarely. We note, for future reference, that the dispersive terms (represented by the derivatives of $\eta_0$ with respect to $\xi$, beyond the first) contribute to both the pressure, (12), and the horizontal velocity component, (13). Further, these terms ($\eta_{0\xi\xi}$) are intimately connected with the $z$-structure of the problem arising, essentially, because $p_z$ is no longer zero at this order. Finally, although the equation for $\eta_0$ is completely determined (equation (11)), the term $\eta_1$ is arbitrary at this stage of the calculation; the equation for $\eta_1$ can be found by continuing the procedure (as we shall mention later).

With these familiar equations and ideas in place, we may now turn to the main issues that we wish to address here.

## 3. The Green–Naghdi equations

In order for us to provide a description of the conventional position of the CH equation within the context of a water–wave model, we first require a reduced version of the Green–Naghdi equations. These equations were developed (Green & Naghdi 1976) within the framework of rather general considerations of continuum mechanics, coupled with one approximation. Here, we present a derivation of the relevant form of the Green–Naghdi (GN) equations, which follows directly from equations (4)–(8), and is the starting point for our discussion of the CH equation.

In our governing equations, (4)–(8), with $\delta$ retained, we assume that $u$ is not a function of $z$. This, we know, is not correct at $O(\varepsilon)$ (see equation (13)), but this approximation is valid for the leading-order problem. This assumption is equivalent to the simplifying approximation used by Green & Naghdi (namely, that $w$ is linear in $z$ in a single-layer model), which these authors are careful to relate to conventional KdV derivations. Thus we have, from (6),

$$w = -zu_x,$$

which satisfies the bottom condition, (8). Then equation (5) leads to

$$p = \eta - \tfrac{1}{2}\delta^2\{(1 + \varepsilon\eta)^2 - z^2\}(u_{xt} + \varepsilon uu_x - \varepsilon u_x^2), \tag{14}$$

which satisfies the pressure condition at the surface. This expression for $p$ is now used in equation (4), which is then integrated over all $z$ to give

$$u_t + \varepsilon uu_x + \eta_x = \frac{\delta^2/3}{(1 + \varepsilon\eta)}\frac{\partial}{\partial x}\left\{(1 + \varepsilon\eta)^3(u_{xt} + \varepsilon uu_{xx} - \varepsilon u_x^2)\right\}. \tag{15}$$

A second equation relating $u$ and $\eta$ is simply the mass conservation equation obtained by also integrating equation (6) over $z$, to produce the familiar result

$$\eta_t + [u(1 + \varepsilon\eta)]_x = 0. \tag{16}$$

Equations (15) and (16) are the GN equations as relevant to one-dimensional wave motion over a flat, horizontal bed. This pair of equations is the usual starting point for a derivation of the CH equation; see, for example, Camassa *et al.* (1994), although we have written our equations in non-dimensional scaled variables and therefore have retained both parameters, $\varepsilon$ and $\delta$

The GN equation, (15), can be obtained by a different but more satisfactory route, although appropriate simplifying assumptions are still required. Formulating the problem for $\eta$ and the average of $u$ ($\bar{u}$, say) taken over the depth (exactly as we introduce in (30) below), Su & Gardner (1969) show that the GN equation, (15), is recovered. This derivation invokes irrotationality and necessitates that higher-order terms are neglected; these arise, for example, when estimates are made for terms such as $\overline{u^2}$. The GN equation that we use, (15), is exact, under the assumption that there is just one layer and that, through it, $w$ is linear in $z$. As we shall describe here, in whatever form we derive them, the GN equations do not lead, systematically, to the CH equation because, we argue, the detailed behaviour through the depth of the fluid is suppressed, and this is essential for a comprehensive description of the problem.

## 4. The Camassa–Holm equation

The conserved energy for the GN equations, (15) and (16), is

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\{(1+\varepsilon\eta)\varepsilon^2u^2 + \tfrac{1}{3}\varepsilon^2\delta^2(1+\varepsilon\eta)^3u_x^2 + \varepsilon^2\eta^2\}\,\mathrm{d}x, \tag{17}$$

for which the variational in $u$ yields

$$\frac{\delta H}{\delta u} = \varepsilon^2(1+\varepsilon\eta)u - \tfrac{1}{3}\varepsilon^2\delta^2\frac{\partial}{\partial x}\left\{(1+\varepsilon\eta)^3u_x\right\}, \tag{18}$$

which we use to define the dynamical variable, $m$:

$$\frac{\delta H}{\delta u} = \varepsilon m, \tag{19}$$

which is written in this form because, later, we must make a selection that requires $m = O(1)$.

From our GN equations, we can show that an equation for $m$ can be obtained by finding, for example, an expression for $m_t$; this gives

$$m_t = -\varepsilon(um_x + 2mu_x) + (1+\varepsilon\eta)\frac{\partial}{\partial x}\left\{-\varepsilon\eta + \tfrac{1}{2}\varepsilon^2u^2 + \tfrac{1}{2}\varepsilon^2\delta^2(1+\varepsilon\eta)^2u_x^2\right\}.$$

Now this equation, together with equation (16), can be written in Lie–Poisson Hamiltonian form as

$$\begin{pmatrix} m \\ 1+\varepsilon\eta \end{pmatrix}_t = -\begin{pmatrix} \partial_x[m + m\partial_x[ & (1+\varepsilon\eta)\partial_x[ \\ \partial_x[(1+\varepsilon\eta) & 0 \end{pmatrix}\begin{pmatrix} H_m] \\ H_h] \end{pmatrix}, \tag{20}$$

where $\partial_x = \partial/\partial x$ and the square brackets have been used to reinforce the interpretation: operate on everything to the right. We may now, in principle, express the Hamiltonian, (17), in terms of $m$ and $\eta$, or $m$ and $u$, as convenient; the former choice yields the variational derivatives $H_m \equiv \delta H/\delta m$ and $H_h \equiv \delta H/\delta h$ as the coefficients, respectively, of

$$\delta H = \int_{-\infty}^{\infty}\{\varepsilon u\delta m + [\varepsilon\eta - \tfrac{1}{2}\varepsilon^2u^2 - \tfrac{1}{2}\varepsilon^2\delta^2(1+\varepsilon\eta)^2u_x^2]\delta h\}\,\mathrm{d}x,$$

where $h = \varepsilon\eta$. This result follows by first using the definition of $m$ in (17) to give

$$H = \frac{1}{2}\int_{-\infty}^{\infty}(\varepsilon u m + \varepsilon^2\eta^2)\,\mathrm{d}x,$$

which can be written as

$$H = \int_{-\infty}^{\infty}\left\{\varepsilon u m - \tfrac{1}{2}\varepsilon^2(1+\varepsilon\eta)u^2 - \tfrac{1}{6}\varepsilon^2\delta^2(1+\varepsilon\eta)^3u_x^2 + \tfrac{1}{2}\varepsilon^2\eta^2\right\}\,\mathrm{d}x.$$

Upon taking the variation in $H$ (in terms of $\delta h$, $\delta u$ and $\delta m$), and using the definition of $m$ again, the above expression for $\delta H$ is obtained. (The general description that we use here follows quite closely the presentation in Camassa *et al.* (1994).)

The development hereafter requires the assumption that the given Hamiltonian be expressed in terms of $m$ and $\eta$ (or $m$ and $u$), and that we dispense altogether with the defining relation implied by (18) and (19). With this interpretation in mind, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(\sqrt{m}-1)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty}\frac{m_t}{\sqrt{m}}\,\mathrm{d}x,$$

provided that both these integrals exist; this can be written, from equation (20), as

$$-\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{m}}\left\{\frac{\partial}{\partial x}(\varepsilon u m) + m\frac{\partial}{\partial x}(\varepsilon u) + (1+\varepsilon\eta)\frac{\partial}{\partial x}(H_h)\right\}\,\mathrm{d}x.$$

We integrate by parts and invoke decay conditions at infinity ($u \to 0$ and $H_h \to 0$ as $|x| \to \infty$), to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(\sqrt{m}-1)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty}H_h\frac{\partial}{\partial x}\left(\frac{1+\varepsilon\eta}{\sqrt{m}}\right)\,\mathrm{d}x.$$

Thus, if $1+\varepsilon\eta \propto \sqrt{m}$ (the proportionality being, at most, a function of $t$), then $\int_{-\infty}^{\infty}(\sqrt{m}-1)\,\mathrm{d}x$ is a constant of the motion (and for any $H_h$ which decays at infinity, we observe, provided that $u$ also decays). The conventional Camassa–Holm approach makes the assumption that this integral is indeed a constant of the motion. In particular, let us write

$$1+\varepsilon\eta = \sqrt{m} = \sqrt{1+\varepsilon\mu}, \tag{21}$$

with $\eta \to 0$ and $\mu \to 0$ (so $m \to 1$) as $|x| \to \infty$; equation (21) then selects a submanifold of the Hamiltonian which describes those solutions which are right-running waves and which satisfy the assumed form embodied in (21). (We shall write more of submanifolds, and the correct submanifold for the GN equations, later.)

The conflict between the choice for $m$ given in (21), and that implied by (18) and (19), is not addressed in the various descriptions of this approach, the argument being that (18) is set aside before we reach (20) and (21). We comment, however, that one possible device that appears to afford some measure of consistency is to impose an irrotational condition on the flow. This requires that $u_z - \delta^2 w_x = 0$ and then equation (4) implies that $\int_{-\infty}^{\infty}u\,\mathrm{d}x$ is a constant of the motion; this follows by substituting for $u_z$ into (4), integrating over all $x$ and imposing undisturbed, uniform conditions at infinity. If, now, the additional term $2\varepsilon u$ is included in the Hamiltonian, (17), we find that $\delta H/\delta u = \varepsilon(1+m)$ (and then we may work with $1+m$, $m = O(\varepsilon)$, rather than $m$).

Nevertheless we proceed, and then on the submanifold given by (21), the Hamiltonian becomes

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\varepsilon^2u^2\sqrt{1+\varepsilon\mu} + \tfrac{1}{3}\varepsilon^2\delta^2u_x^2(1+\varepsilon\mu)^{3/2} + \left(\sqrt{1+\varepsilon\mu}-1\right)^2\right\}\mathrm{d}x, \tag{22}$$

where $H$ is now expressed in terms of $\mu$ (i.e. $m$) and $u$.

To complete the calculation, we follow the idea developed by Olver (1988) (and explained more fully in Camassa *et al.* 1994) by first expanding $\mu$ (in $\varepsilon$) and then finding the variations in $u$ at each order, where

$$\frac{\delta H}{\delta u} = \varepsilon(1 + \varepsilon\mu),$$

and equation (18) is hereafter ignored. This definition, coupled with equation (22), determines $\mu$ in terms of $u$.

Thus, with

$$\mu \sim \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2,$$

we find that

$$\tfrac{1}{2}D\mu_0 = 1,$$

where $D$ is the variational derivative, and hence

$$\mu_0 = 2u + M,$$

where $M$ is any term which vanishes on taking the variational derivative, e.g. a term $\partial^n u/\partial x^n$; let us write

$$\mu_0 = 2u - a\delta^2 u_{xx},$$

where $a$ is an arbitrary function of $t$; Camassa *et al.* (1994) then retain terms as far as $O(\varepsilon^2)$, i.e. $\mu_2$, but eventually work with $\mu$ only as far as $\mu_0$. The resulting (approximate) equation of motion is

$$m_t \sim -\left\{\frac{\partial}{\partial x}(mH_m) + m\frac{\partial H_m}{\partial x}\right\},$$

(since $H_h = O(\varepsilon^2)$ on the submanifold); the requirement that the higher-order terms in the expansion of the Hamiltonian, (22), are also conserved by the flow represented by the $m$ which satisfies the approximate equation, leads to $a = \frac{2}{3}$. Now we see that

$$H \sim \frac{1}{2}\int_{-\infty}^{\infty} \left(\varepsilon\mu + \tfrac{1}{2}\varepsilon^2 u\mu\right) \mathrm{d}x$$

and then Camassa *et al.* (1994) use this to give

$$H_m \equiv \frac{\delta H}{\delta m} = \frac{\delta H}{\delta(\varepsilon\mu)} \sim \tfrac{1}{2} + \tfrac{1}{2}\varepsilon u,$$

(the argument being, presumably, that $\frac{1}{4}\varepsilon^2 u\mu$ is essentially $\frac{1}{8}\varepsilon^2\mu^2$, for which the variational is $\frac{1}{4}\varepsilon\mu$ and then, by the same token, this is replaced by $\frac{1}{2}\varepsilon u$, which certainly follows if we invoke $\delta \to 0$).

This expression for $H_m$, together with

$$m \sim 1 + \varepsilon(2u - \tfrac{2}{3}\delta^2 u_{xx}) \tag{23}$$

gives the equation for $u(x, t)$, from the equation of motion, as

$$u_t + u_x + \tfrac{3}{2}\varepsilon u u_x - \tfrac{1}{6}\delta^2 u_{xxx} - \tfrac{1}{3}\delta^2 u_{xxt} = \tfrac{1}{6}\varepsilon\delta^2(2u_x u_{xx} + u u_{xxx}) + O(\varepsilon^2), \tag{24}$$

as $\varepsilon \to 0$ at fixed $\delta$. This is the Camassa–Holm equation, although it is more commonly written in the form

$$u_t + 2\kappa u_\zeta + \tfrac{3}{2}\varepsilon u u_\zeta - \tfrac{1}{3}\delta^2 u_{\zeta\zeta t} = \tfrac{1}{6}\varepsilon\delta^2(2u_\zeta u_{\zeta\zeta} + u u_{\zeta\zeta\zeta}), \tag{25}$$

where $\zeta = x - \frac{1}{2}t$, $\kappa = \frac{1}{4}$ here, and $\varepsilon$ and $\delta$ are usually ignored (by scaling them out or, equivalently, setting $\varepsilon = \delta = 1$). The frame represented by the choice of $\zeta$ is simply that in which $u_{xxx}$ is removed in favour of $u_{xxt}$. The use of this frame, however, implies that $\kappa \neq 0$, although the selection $\kappa = 0$ is often made in studies of the CH equation (see Camassa & Holm 1993; Camassa *et al.* 1994; Fisher & Schiff 1999).

## 5. Comments on the CH derivation

It is clear that the derivation of the Camassa–Holm equation leaves much to be desired. In particular, the approach rehearsed here does not follow a consistent mathematical development. Further, it is not clear whether the details of the $z$-structure are relevant; certainly our equations use only average properties, in some sense, and so the variation with depth of the dispersive contributions, for example, is lost. This is compounded by the choice of a special submanifold which is clearly not correct (even at this order of approximation) for the GN equations. (The usual choice of $\kappa = 0$ is not so significant, for the CH equation is completely integrable for all $\kappa$; $\kappa = 0$ can be regarded merely as a device for obtaining some simple results and so, perhaps, allowing us an insight into the nature of the problem.) We will first remove the second assumption – the choice of submanifold – and hence obtain the appropriate unidirectional form of the GN equations.

As Olver (1988) points out, and as is self-evident, the correct choice of submanifold is the one which recovers the solution that is obtained directly from the governing equations. We wish to seek an asymptotic solution of equations (15) and (16) in the form

$$u \sim u_0 + \varepsilon u_1, \quad \eta \sim \eta_0 + \varepsilon \eta_1,$$

at fixed $\delta$. The leading order is then given by

$$u_{0tt} - u_{0xx} = \tfrac{1}{3}\delta^2 u_{0xxtt}, \quad \eta_{0t} + u_{0x} = 0,$$

which represents a linear, dispersive wave. The familiar approach in these problems is to consider unidirectional propagation which is non-dispersive, to leading order, in some region of physical space. Here, if we are in the region defined by $(x, t) = O(1)$, then we must select $\delta$ small; on the other hand, for arbitrary (fixed) $\delta$ – our preferred choice – we must be in an appropriate far field defined by a suitable scaling of $(x, t)$. In either case, the model is equivalent to supposing that the waves are long.

The former choice yields, from equations (15) and (16),

$$u_t + u_x + \tfrac{3}{2}\varepsilon u u_x - \tfrac{1}{12}\delta^2 u_{xxx} - \tfrac{1}{4}\delta^2 u_{xxt} = -\tfrac{1}{24}\varepsilon\delta^2(14u_x u_{xx} + u u_{xxx}) + O(\varepsilon^2, \delta^4), \qquad (26a)$$

which can be recast as a CH equation by moving in the frame $(x - \frac{1}{3}t)$ (which eliminates the term $u_{xxx}$), and then $\kappa = \frac{1}{3}$ (see equation (25)). It is clear, however, that equation (26*a*) can be written in a number of alternative forms, all valid at this level of approximation. Thus we can use

$$u_t \sim -(u_x + \tfrac{3}{2}\varepsilon u u_x)$$

in the term $u_{xxt}$ and so produce

$$u_t + u_x + \tfrac{3}{2}\varepsilon u u_x + \frac{\delta^2}{6}u_{xxx} = -\frac{\varepsilon\delta^2}{24}(41u_x u_{xx} + 10u u_{xxx}) + O(\varepsilon^2, \delta^4). \qquad (26b)$$

However, this equation now lacks the term $u_{xxt}$ but, as we will show, this is easily remedied.

The corresponding submanifold for this problem (which can be regarded as the relation which determines $\eta$, given $u$, for right-running waves) is

$$\eta \sim u + \tfrac{1}{4}\varepsilon u^2 + \tfrac{1}{2}\delta^2(u_{xt} - u_{xx}) - \tfrac{1}{24}\varepsilon\delta^2\left(7uu_{xx} + \tfrac{11}{2}u_x^2\right). \tag{27}$$

On the other hand, the submanifold used to obtain the CH equation is

$$1 + \varepsilon\eta = \sqrt{1 + \varepsilon\mu} \quad \text{with} \quad \mu \sim 2u - \tfrac{2}{3}\delta^2 u_{xx},$$

i.e.

$$\eta \sim u - \tfrac{1}{2}\varepsilon u^2 - \tfrac{1}{3}\delta^2 u_{xx} + \tfrac{1}{3}\varepsilon\delta^2 uu_{xx}. \tag{28}$$

The two results, (27) and (28), agree only as far as the first term. (Correspondingly, the submanifold for left-going waves is, to leading order, $\eta \sim -u$; we could equally elect to follow this wave.)

The second and more attractive alternative is to scale to generate a far field (defined, for example, by $(x, t) = O(\varepsilon^{-1/2})$, $\delta$ fixed). This procedure recovers precisely the equations (26$a, b$), with $x$ and $t$ suitably redefined and $\delta^2$ replaced by $\varepsilon\delta^2$; the error is now simply $O(\varepsilon^3)$.

The most notable property of our equations for $u$, (26$a, b$), is the form of the nonlinear dispersive terms (the right-hand side): no scaling property exists (but see below) which will produce the right-hand side of equation (25), i.e. coefficients in the ratio 2 : 1. This particular relation is critical for complete integrability of the equation. In summary, a device has been invented – the choice of submanifold – which engineers the appearance of a completely integrable equation from the GN equations. Let us now briefly examine what happens when we dispense with the GN equations and work from the original set of governing equations, (4)–(8).

The most natural way to proceed is simply to extend the derivation of the KdV equation, (11). Thus, we replace $\delta^2$ by $\varepsilon$ (according to the transformation (9), but this will be relaxed later), introduce $\xi = \chi - \theta$, $\tau = \varepsilon\theta$, and seek an asymptotic solution (for $\varepsilon \to 0$)

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n \quad (q \equiv u, w, p, \eta).$$

When we find the complete solution correct at $O(\varepsilon)$, the surface profile satisfies

$$2\eta_\tau + 3\eta\eta_\xi + \tfrac{1}{3}\eta_{\xi\xi\xi} - \tfrac{3}{4}\varepsilon\,\eta^2\eta_\xi + \tfrac{19}{180}\varepsilon\,\eta_{\xi\xi\xi\xi\xi} = -\tfrac{1}{12}\varepsilon\,(23\eta_\xi\eta_{\xi\xi} + 10\eta\eta_{\xi\xi\xi}) + O(\varepsilon^2). \tag{29}$$

Correspondingly, if we define the mean horizontal velocity component, $\bar{u}$, by

$$\int_0^{1+\varepsilon\eta} u(x, t, z; \varepsilon)\,\mathrm{d}z = (1 + \varepsilon\eta)\bar{u}(x, t; \varepsilon), \tag{30}$$

then $\bar{u}$ satisfies

$$2\bar{u}_\tau + 3\bar{u}\bar{u}_\xi + \tfrac{1}{3}\bar{u}_{\xi\xi\xi} + \tfrac{19}{180}\varepsilon\,\bar{u}_{\xi\xi\xi\xi\xi} = -\tfrac{1}{12}\varepsilon(41\bar{u}_\xi\bar{u}_{\xi\xi} + 10\bar{u}\bar{u}_{\xi\xi\xi}) + O(\varepsilon^2). \tag{31}$$

This corresponds, in the absence of the fifth-derivative term, with equation (26$b$) and confirms the version of the GN equations obtained by Su & Gardner (1969). Those equations are precisely our GN equations, with $\bar{u}$ written for $u$; Su & Gardner's approach, based on a depth-average of the full equations, omits only the higher-order linear dispersive term. (We see that, indeed, $u = \bar{u}$ by virtue of equation (30), when the single-layer model with $u = u(x, t; \varepsilon)$ is used.)

The submanifold in this case is represented by

$$\eta \sim \bar{u} + \tfrac{1}{4}\varepsilon\bar{u}^2 - \tfrac{1}{6}\varepsilon\,\bar{u}_{\chi\chi} - \varepsilon^2\left(\tfrac{17}{48}\bar{u}\bar{u}_{\chi\chi} + \tfrac{61}{180}\bar{u}_\chi^2 + \tfrac{51}{240}\bar{u}_{\chi\chi\chi\chi}\right), \tag{32}$$

where we have reverted to the original scaled variables, $(\chi, \theta)$. In the context of the water-wave problem, and working consistently with terms $O(\varepsilon)$ for arbitrary $\delta$ (because the $\chi$ and $\theta$ used here have been scaled according to (9)), equation (31) is the 'correct' equation (in that we have derived it directly from the full governing equations). The obvious question now is whether it is possible to transform equation (31) into a CH equation, at least, if we ignore the fifth-derivative term.

Equation (31), written in terms of $\chi$ and $\theta$, becomes

$$\bar{u}_\theta + \bar{u}_\chi + \tfrac{3}{2}\varepsilon\bar{u}\bar{u}_\chi + \tfrac{1}{6}\varepsilon\bar{u}_{\chi\chi\chi} + \tfrac{19}{360}\varepsilon^2\,\bar{u}_{\chi\chi\chi\chi\chi} = -\tfrac{1}{24}\varepsilon^2(41\bar{u}_\chi\bar{u}_{\chi\chi} + 10\bar{u}\bar{u}_{\chi\chi\chi}) + O(\varepsilon^3), \tag{33}$$

to which we may add (on the left, say)

$$\varepsilon\mu\bar{u}_{\chi\chi\theta} - \varepsilon\mu\bar{u}_{\chi\chi\theta}, \tag{34}$$

for arbitrary real $\mu$, and use (in the first term, say)

$$\bar{u}_\theta \sim -(\bar{u}_\chi + \tfrac{3}{2}\varepsilon\bar{u}\bar{u}_\chi),$$

if we do hereafter ignore the fifth-derivative term (and so we drop the term $-\frac{1}{6}\varepsilon\bar{u}_{\chi\chi\chi}$). Then, for the choice $\mu = \frac{7}{12}$, the nonlinear dispersive terms (on the right-hand side) are in the ratio $2:1$; however, the other coefficients cannot be scaled to recover a CH equation. (The CH equation, in the case $\kappa = 0$, must be one of the family

$$u_t + 3\alpha u u_x - \beta u_{xxt} = \alpha\beta(2u_x u_{xx} + u u_{xxx}), \tag{35}$$

where $\alpha \neq 0$ and $\beta > 0$ are real arbitrary constants.) The observation that we are able to adjust the nonlinear dispersive terms is encouraging and something that we shall return to shortly; however, it is clear that, in order to recover the CH equation, we require a little more freedom in choosing the coefficients. At this stage, we conclude that the equation, (33), for $\bar{u}$ is not a CH equation on two counts: the presence of higher-order linear dispersion ($\bar{u}_{\chi\chi\chi\chi\chi}$) and the non-existence of a scaling which will transform the other terms. (Precisely the same conclusion obtains for equations (26*a*) and (26*b*) when this same manoeuvre, (34), is used there.) Although other equations, similar in character to the CH equation, can be obtained from the GN equations, we do not pursue this route here. (See Camassa, Holm & Levermore (1997) and Choi & Camassa (1999) for some interesting ideas along these lines as they apply to wave propagation in, and on the surface of, a fluid.) We do not follow this path because we have already indicated that the GN equations are, through their suppression of the $z$-structure, a questionable starting point.

One final observation: the equation for $\bar{u}$ (or $u$, via the GN equations) contains only quadratic nonlinearity, whereas the equation for the surface wave, $\eta$, incorporates cubic nonlinearity. This term is eliminated between (29) and (31) by virtue of the nonlinear term, $\frac{1}{4}\varepsilon\bar{u}^2$, in the submanifold expression, (32). Thus, a solution for $\bar{u}$ (if such can be found), which involves quadratic nonlinearity, generates a solution for $\eta$ (via (32)) which satisfies an equation with cubic nonlinearity.

## 6. A Camassa–Holm equation for water waves

We return to our governing equations, (4)–(8), retain both $\varepsilon$ and $\delta$, and scale only with respect to $\varepsilon$; then, for right-running waves, we introduce the appropriate far field defined in terms of $\varepsilon$:

$$\zeta = \sqrt{\varepsilon}(x - t), \quad T = \varepsilon^{3/2}t, \quad w = \sqrt{\varepsilon}W, \tag{36}$$

where we use the symbols $(\zeta, T)$ here (cf. (9) and (10)) because $\delta$ has been retained in the governing equations; the coordinates $(\xi, \tau)$, given in (10), already subsume $\delta$ for they were defined after the application of the scaling transformation, (9).

The governing equations are therefore

$$\left.\begin{aligned}
&\varepsilon u_T - u_\zeta + \varepsilon(uu_\zeta + Wu_z) = -p_\zeta,\\
&\varepsilon\delta^2\{\varepsilon W_T - W_\zeta + \varepsilon(uW_\zeta + WW_z)\} = -p_z,\\
&u_\zeta + W_z = 0,\\
\text{with}\quad &\\
&p = \eta, \quad W = \varepsilon\eta_T - \eta_\zeta + \varepsilon u\eta_\zeta \quad \text{on} \quad z = 1 + \varepsilon\eta,\\
&W = 0 \quad \text{on} \quad z = 0.
\end{aligned}\right\} \tag{37}$$

In this new approach, we seek a solution of the set (37) as a double asymptotic expansion in $\varepsilon$ and $\delta$:

$$q \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \delta^{2m} q_{nm}, \tag{38}$$

as $\varepsilon \to 0$ and $\delta \to 0$, independently; here, $q$ (and correspondingly $q_{nm}$) stands for $u, W, p$ and $\eta$. We follow the procedure that has already given us equations (11) and (29); in this case we obtain (for $T = O(1)$, $\zeta = O(1)$)

$$2\eta_T + 3\eta\eta_\zeta + \tfrac{1}{3}\delta^2\eta_{\zeta\zeta\zeta} - \tfrac{3}{4}\varepsilon\eta^2\eta_\zeta = -\tfrac{1}{12}\varepsilon\delta^2(23\eta_\zeta\eta_{\zeta\zeta} + 10\eta\eta_{\zeta\zeta\zeta}) + O(\varepsilon^2, \delta^4), \tag{39}$$

where $\eta \sim \eta_{00} + \varepsilon\eta_{10} + \delta^2\eta_{01} + \varepsilon\delta^2\eta_{11}$; equation (39), as expected, agrees with equation (29). Note, however, that the fifth-derivative term is now absent: it is $O(\delta^4)$.

At this same order, where we retain terms $O(\varepsilon)$, $O(\delta^2)$ and $O(\varepsilon\delta^2)$, we find that

$$u \sim \eta - \tfrac{1}{4}\varepsilon\eta^2 + \varepsilon\delta^2(\tfrac{1}{3} - \tfrac{1}{2}z^2)\eta_{\zeta\zeta}; \tag{40}$$

it so happens that no term $O(\delta^2)$ arises here. In what follows, it is useful to invert this relation to provide an expression for $\eta$ in terms of $u$ at a specific depth. Let us select $z = z_0$ $(0 \leqslant z_0 \leqslant 1)$ and then write $\lambda = \frac{1}{3} - \frac{1}{2}z_0^2$ $(-\frac{1}{6} \leqslant \lambda \leqslant \frac{1}{3})$. Thus we obtain

$$\eta \sim \hat{u} + \tfrac{1}{4}\varepsilon\hat{u}^2 - \varepsilon\delta^2\lambda\hat{u}_{\zeta\zeta}, \tag{41}$$

where $\hat{u} = u(\zeta, T, z_0; \varepsilon, \delta)$, and we see that (41) agrees with (32) since $\bar{u}$ (the mean) is recovered from the choice $\lambda = \frac{1}{6}$. When we use (41) in (39), we obtain the corresponding equation for $\hat{u}$:

$$2\hat{u}_T + 3\hat{u}\hat{u}_\zeta + \tfrac{1}{3}\delta^2\hat{u}_{\zeta\zeta\zeta} = -\varepsilon\delta^2\{(6\lambda + \tfrac{29}{12})\hat{u}_\zeta\hat{u}_{\zeta\zeta} + \tfrac{5}{6}\hat{u}\hat{u}_{\zeta\zeta\zeta}\} + O(\varepsilon^2, \delta^4). \tag{42}$$

(Again, this checks with equation (31) for $\lambda = \frac{1}{6}$.) An important new ingredient is now evident in our formulation of the problem: we have a free parameter ($\lambda$) which may be chosen so that equation (42) becomes a Camassa–Holm equation (if this is possible); see equation (35). However, even with the inclusion of scaling transformations on $\hat{u}$, $\zeta$ and $T$, this cannot be done. This is only to be expected as it is clear that equation (42) is not of the form (35) for there is no term that corresponds to $u_{xxt}$.

To proceed, let us revert to the original variables, equivalent to $(x, t)$, that we have implied by the $\varepsilon$-only scaling. Thus, from (36), with $\Theta = \sqrt{\varepsilon}t$, $Z = \sqrt{\varepsilon}x$, we have

$$\frac{\partial}{\partial\zeta} \equiv \frac{\partial}{\partial Z}, \quad \varepsilon\frac{\partial}{\partial T} \equiv \frac{\partial}{\partial Z} + \frac{\partial}{\partial\Theta};$$

it is therefore convenient to multiply equation (42) by $\varepsilon$ and then use this transformation. Thus we obtain

$$2(\hat{u}_\Theta + \hat{u}_Z) + 3\varepsilon\hat{u}\hat{u}_Z + \tfrac{1}{3}\varepsilon\delta^2\hat{u}_{ZZZ} = -\varepsilon^2\delta^2\{(6\lambda + \tfrac{29}{12})\hat{u}_Z\hat{u}_{ZZ} + \tfrac{5}{6}\hat{u}\hat{u}_{ZZZ}\} + O(\varepsilon^3, \varepsilon\delta^4), \quad (43)$$

and, in view of equation (35) (see also (34)), we add

$$\varepsilon\delta^2\mu\hat{u}_{ZZ\Theta} - \varepsilon\delta^2\mu\hat{u}_{ZZ\Theta}$$

to the left-hand side of this equation, where $\mu$ is an arbitrary (real) constant. Further, in the first term here, say, we use (43) in the form

$$\hat{u}_\Theta \sim -(\hat{u}_Z + \tfrac{3}{2}\varepsilon\hat{u}\hat{u}_Z) \quad (44)$$

to give

$$2(\hat{u}_\Theta + \hat{u}_Z) + 3\varepsilon\hat{u}\hat{u}_Z + \varepsilon\delta^2(\tfrac{1}{3} - \mu)\hat{u}_{ZZZ} - \varepsilon\delta^2\mu\hat{u}_{ZZ\Theta}$$
$$= -\varepsilon^2\delta^2\{(6\lambda - \tfrac{9}{2}\mu + \tfrac{29}{12})\hat{u}_Z\hat{u}_{ZZ} + (\tfrac{5}{6} - \tfrac{3}{2}\mu)\hat{u}\hat{u}_{ZZZ}\} + O(\varepsilon^3, \varepsilon\delta^4). \quad (45)$$

The CH equation can now be recovered, as we shall demonstrate, by making suitable choices. First, to ensure that the coefficients of $\hat{u}_Z\hat{u}_{ZZ}$ and $\hat{u}\hat{u}_{ZZZ}$ are in the ratio 2 : 1, respectively, we require

$$\mu = \tfrac{1}{2} + 4\lambda. \quad (46)$$

Secondly, the $(\alpha, \beta)$-property satisfied by equation (35) requires that

$$\tfrac{1}{2}\varepsilon\,\tfrac{1}{2}\varepsilon\delta^2\mu = \tfrac{1}{2}\varepsilon^2\delta^2(\tfrac{3}{2}\mu - \tfrac{5}{6})$$

and so $\mu = \frac{5}{6}$, which gives (from equation (46)) $\lambda = \frac{1}{12}$.

Thus equation (45) becomes

$$2(\hat{u}_\Theta + \hat{u}_Z) + 3\varepsilon\hat{u}\hat{u}_Z - \tfrac{1}{2}\varepsilon\delta^2\hat{u}_{ZZZ} - \tfrac{5}{6}\varepsilon\delta^2\hat{u}_{ZZ\Theta} = \tfrac{5}{12}\varepsilon^2\delta^2(2\hat{u}_Z\hat{u}_{ZZ} + \hat{u}\hat{u}_{ZZZ}) + O(\varepsilon^3, \varepsilon\delta^4),$$

which can be written in the standard CH form by moving in the frame defined by $X = Z - \frac{3}{5}\Theta$ (so that the term $\hat{u}_{ZZZ}$ is removed in favour of $\hat{u}_{ZZ\Theta}$) and then, introducing the scaling transformation

$$X \to \tfrac{1}{2}\sqrt{\tfrac{5}{3}}X, \quad \hat{u} \to \sqrt{\tfrac{5}{3}}\hat{u} \quad (\text{and } \Theta \to \Theta),$$

(simply to recast the equation precisely in the form (35)), this gives

$$\hat{u}_\Theta + \tfrac{4}{5}\sqrt{\tfrac{3}{5}}\hat{u}_X + 3\varepsilon\hat{u}\hat{u}_X - \varepsilon\delta^2\hat{u}_{XX\Theta} = \varepsilon^2\delta^2(2\hat{u}_X\hat{u}_{XX} + \hat{u}\hat{u}_{XXX}) + O(\varepsilon^3, \varepsilon\delta^4). \quad (47)$$

Equation (47) is a Camassa–Holm equation, with error $O(\varepsilon^3, \varepsilon\delta^4)$ and with $2\kappa = \frac{4}{5}\sqrt{\frac{3}{5}}$; see equation (3). (Of course, precise equivalence necessitates, formally, dropping the error terms and then either setting $\varepsilon = \delta = 1$ or applying a further scaling transformation to remove $\varepsilon$ and $\delta$, i.e. $(X, \Theta) \to \delta\sqrt{\varepsilon}(X, \Theta)$, $\hat{u} \to \hat{u}/\varepsilon$.) This equation describes the horizontal velocity component at a certain depth in the fluid; with $\lambda = \frac{1}{12}$ this corresponds to $z_0 = 1/\sqrt{2}$. Thus, solving equation (47) (which, because $\kappa \neq 0$, will not have peakon solutions) for $\hat{u}$, the surface wave (from (41)) is given by

$$\eta = \sqrt{\tfrac{5}{3}}\left(\hat{u} + \tfrac{1}{4}\sqrt{\tfrac{5}{3}}\varepsilon\hat{u}^2 - \tfrac{1}{5}\varepsilon\delta^2\hat{u}_{XX}\right) + O(\varepsilon^2, \delta^4). \quad (48)$$

Indeed, to this order, we can reverse the process by using the resulting expression for $\eta$ in (40) and thereby obtain $u(\zeta, T, z; \varepsilon, \delta)$ which, for selected $z$ ($\neq 1/\sqrt{2}$), will be a solution of an equation of general Camassa–Holm-type, but not of the precise form (35).

It should be noted that it is the horizontal velocity component at a specific depth which is described by a CH equation, not the surface wave, nor some averaged horizontal velocity component. A representation of the surface in our formulation is obtained via the nonlinear mapping, (48), or, equivalently, directly from equation (39). This equation, we observe, is an extension of the KdV equation, which is itself recovered as we let $\varepsilon \to 0$ at $\delta$ fixed. These results appear to correspond to those quoted by Fokas (1995), although the connection with the CH equation is not developed there. Further, it would seem that the analysis owes much to the GN approach since the $z$-dependence appears to have been simplified. Certainly, the importance of the horizontal velocity component at a specific depth is not discussed. We do comment, however, that the introduction of a specific depth recalls one of the methods used to extend the Boussinesq equation (in the context of water waves), see, e.g. Madsen & Shaeffer (1999).

## 7. Some results

We start with equation (47), which we will write in the form

$$\hat{u}_\Theta + 2\kappa\hat{u}_X + 3\varepsilon\hat{u}\hat{u}_X - \varepsilon\delta^2\hat{u}_{XX\Theta} = \varepsilon^2\delta^2(2\hat{u}_X\hat{u}_{XX} + \hat{u}\hat{u}_{XXX}), \tag{49}$$

where the error $O(\varepsilon^3, \varepsilon\delta^4)$ is understood and $\kappa = \frac{2}{5}\sqrt{\frac{3}{5}}$. This equation possesses solitary-wave and soliton solutions for all $\kappa$; the soliton solutions, spectral properties and inverse scattering are described in Camassa *et al.* (1994) and in Constantin (2001). Of course, the case $\kappa = 0$ is not relevant to the water-wave problem, but we could use it as a simple indicator of what is possible. (The great advantage of this case is that the solitary-wave and soliton solutions take particularly simple exact forms.)

The solitary-wave solution of equation (49) is a special travelling-wave solution:

$$\hat{u}(X, \Theta) = f(X - c\Theta),$$

where $c$ is a (real) constant; on the assumption that $f$, $f'$ and $f''$ all tend to zero as $|X - c\Theta| \to \infty$ (so that periodic solutions are excluded), we obtain

$$\varepsilon\delta^2(f')^2 = f^2\frac{(c - 2\kappa - \varepsilon f)}{(c - \varepsilon f)}. \tag{50}$$

(The prime denotes the derivative with respect to $X - c\Theta$.) Sadly, for arbitrary $\kappa$ and $\varepsilon$, it is not possible to integrate (50) and write the solution, explicitly, in closed form. However, the special case $\kappa = 0$ evidently produces a simple result:

$$f = a\exp(-|X - c\Theta|/\delta\sqrt{\varepsilon}), \tag{51}$$

and, if we satisfy the jump condition (across $X = c\Theta$) obtained by integrating equation (49) once with respect to $(X - c\Theta)$, then we require $a = c$. The function (51), with $a = c$, is the peakon solution (although, of course, this is not a proper solution of the equation). For arbitrary $\kappa$ (but clearly we require $c \geqslant 2\kappa$ for solutions to exist), equation (50) can be integrated (Camassa *et al.* 1994) to give $f$ implicitly:

$$\left(\frac{F - \gamma}{F + \gamma}\right)^\gamma \left(\frac{F + 1}{F - 1}\right) = \exp(-(X - c\Theta)/\delta\sqrt{\varepsilon}) \tag{52}$$

where

$$F = \sqrt{\frac{c - \varepsilon f}{c - 2\kappa - \varepsilon f}}, \quad \gamma = \sqrt{\frac{c}{c - 2\kappa}}, \tag{53}$$

ignoring the phase-shift in $X$ (as we have also done in (51)), which would simply replace $X$ by $X+\text{constant}$.

In the context of our water-wave problem, and with the errors implied by using equation (49), we may approximate (52) and (53) by allowing $\varepsilon \to 0$. With this choice, $c = O(1)$ and selecting $c \geqslant 2\kappa$, then either from equations (52) and (53), or directly from equation (50), we find that

$$\hat{u} = f \sim a\,\mathrm{sech}^2\left\{\beta(X - c\Theta)\right\}, \tag{54}$$

where

$$\beta \sim \frac{1}{2\delta}\sqrt{\frac{a}{2\kappa}}\left(1 - \frac{\varepsilon a}{2\kappa}\right)$$

and $c \sim 2\kappa + \varepsilon a; a\,(>0)$ is the arbitrary amplitude of the wave. The surface wave, $\eta$, is then obtained from (48); written in our $(Z,\Theta)$ variables, this is

$$\eta \sim A\,\mathrm{sech}^2\left\{\alpha(Z - \Theta - \tfrac{1}{2}\varepsilon A\Theta)\right\} + \tfrac{5}{8}\varepsilon A^2\mathrm{sech}^4\left\{\alpha(Z - \Theta)\right\}, \tag{55}$$

where

$$\alpha \sim \frac{1}{2\delta}\sqrt{3A}(1 - \tfrac{9}{8}\varepsilon A)$$

and

$$A \sim a\sqrt{\tfrac{5}{3}}\left(1 - \tfrac{1}{4}\varepsilon a\sqrt{\tfrac{5}{3}}\right).$$

Thus expression (55) is the solution (written in terms of $(Z,\Theta)$; see (36)) of equation (39), valid as $\varepsilon \to 0$. To leading order, this recovers the classical solitary-wave solution with amplitude $A$. Of some interest is the observation that a single solitary-wave-type solution ($\mathrm{sech}^2$) for $\hat{u}$ generates both the $\mathrm{sech}^2$ solution for $\eta$, and its perturbation ($\mathrm{sech}^4$). Conversely, if we take this solution for $\eta$ and use it in equation (40) for $u$, the choice $\lambda = \frac{1}{3} - \frac{1}{2}z_0^2 = \frac{1}{12}$ is the only one which eliminates the $\mathrm{sech}^4$ term, producing a solution which is purely $\mathrm{sech}^2$ (to this order) for $\hat{u}$. That is, for other choices of $z_0$ ($0 \leqslant z_0 \leqslant 1, z_0 \neq 1/\sqrt{2}$), the horizontal velocity component contains both $\mathrm{sech}^2$ and $\mathrm{sech}^4$ contributions, to this order.

In passing, we comment that the corresponding discussion which starts from the peakon solution, (51), generates the associated surface wave (from (48)) and then uses this to investigate Camassa–Holm-type equations for $\hat{u}(z \neq z_0)$ is futile. Thus, for example, (48) yields

$$\eta \sim \sqrt{\tfrac{5}{3}}\{\tfrac{4}{5}c\exp(-|X - c\Theta|/\delta\sqrt{\varepsilon}) + \tfrac{1}{4}\varepsilon\sqrt{\tfrac{5}{3}}c^2\exp(-2|X - c\Theta|/\delta\sqrt{\varepsilon})\},$$

but then the function

$$u = a\exp(-|X - c\Theta|/\delta\sqrt{\varepsilon}) + \varepsilon b\exp(-2|X - c\Theta|/\delta\sqrt{\varepsilon}),$$

for suitable $a$ and $b$, is not a solution of the more general equation that arises for $\hat{u}$ with $z_0 \neq 1/\sqrt{2}$. It is straightforward to confirm that the only solution of this type that exists is where $\mu = \frac{5}{6}$, $\lambda = \frac{1}{12}$ ($z_0 = 1/\sqrt{2}$) and then $b = 0$. That difficulties are encountered is to be expected: we have simply set $\kappa = 0$, which is inconsistent with the set of equations that describe this water-wave problem.

One final, but significant, observation is that the procedure which has given us equation (39) for $\eta$ and, eventually, equation (47) for $\hat{u}$, may not produce a uniformly valid approximation. This becomes clear when we carefully examine equation (47); for example, the term $O(\varepsilon\delta^4)$ is precisely the fifth-derivative term that appears in our extended KdV equation, (29). Now, if we have initial data which are consistent with the scalings leading to equation (47), then an initial profile will evolve (at this order) according to this equation. However, if this solution exhibits, for example, a steepening of the waveform (which is known to occur with the CH equation; see Camassa *et al.* 1994; Constantin & Escher 1998), then in an $O(\delta)$ neighbourhood of where this occurs (say, at $(X_0, \Theta_0)$), so that

$$X - X_0 = O(\delta), \quad \Theta - \Theta_0 = O(\delta),$$

we see that the terms in equation (47) are $O(\delta^{-1})$, $O(\varepsilon\delta^{-1})$ or $O(\varepsilon^2\delta^{-1})$, and the omitted fifth-derivative term becomes $O(\varepsilon\delta^{-1})$. This result is no more than a re-statement of the $\delta$-scaling property of the original equations, (4)–(8), and leads to the recovery of the extended KdV equation, (29). A reasonable interpretation of this property is: whereas the third derivative inhibits the breaking associated with the nonlinear term in the classical KdV equation, the fifth derivative becomes available to counteract the singularities that may be generated by the nonlinear dispersive terms. Thus, on sufficiently short scales measured by $\delta$ and for suitable initial data, the Camassa–Holm approximation appears to fail. This non-uniformity is the mechanism by which steepening (or perhaps sharp-peaked or breaking) solutions of CH, which might be regarded as unsatisfactory, are, we presume, corrected and thereby may become more acceptable solutions for the water-wave problem. Certainly, however the solution may evolve or what elements are thought to be relevant, all this adds to the richness of the behaviours that are accessible through even the simplest model for water waves.

For completeness, we record that the corresponding solution of the extended KdV equation, (29), is

$$\eta \sim a\,\mathrm{sech}^2\{\beta(\xi - c\tau)\} + \varepsilon\tfrac{27}{16}a^2\mathrm{sech}^4\{\beta(\xi - c\tau\}$$

where

$$c \sim \tfrac{1}{2}a + \tfrac{71}{30}\varepsilon a^2$$

and

$$\beta \sim \sqrt{\tfrac{3a}{4}}\,(1 + \tfrac{227}{120}\varepsilon a).$$

This has the same structure as solution (55) obtained via the Camassa–Holm route: the coefficient of the fifth-derivative term merely contributes to the $O(\varepsilon)$ terms throughout, adjusting the coefficients accordingly.

## 8. Camassa–Holm in two dimensions

Now that we have in place a systematic derivation of the Camassa–Holm equation, which demonstrates its validity within the water-wave problem, we may, with some confidence, explore its role in other water-wave models. Thus, for example, we could investigate how the CH equation appears in the problem of waves propagating over an arbitrary 'shear' flow, or in two dimensions; we choose to develop (as an example of what is possible) a CH equation in two dimensions. The resulting equation – in fact, more than one variant exists – will be the CH counterpart of the two-dimensional KdV equation, often referred to as the Kadomtsev–Petviashvili (KP) equation.

The governing equations are (4)–(8), with the $y$-dependence added and the velocity component in this direction written as $v$. Both $v$ and $y$ are non-dimensionalized exactly as for $u$ and $x$; we retain $\delta$ and then scale $y$ (and correspondingly $v$) with respect to $\varepsilon$, in keeping with the approach usually adopted for the KP equation. In this case, we scale

$$y = Y/\sqrt{\varepsilon}, \quad v = \sqrt{\epsilon}V; \tag{56}$$

however, because the CH equation retains terms as far as $O(\varepsilon)$, an alternative (and simpler) two-dimensional CH equation is generated by the scaling

$$y = Y/\varepsilon, \quad v = \varepsilon V. \tag{57}$$

In either case, the resulting equations are

$$u_t + \varepsilon(uu_x + \Delta V u_Y + wu_z) = -p_x,$$
$$V_t + \varepsilon(uV_x + \Delta V V_Y + wV_z) = -p_Y,$$
$$\varepsilon\delta^2\{w_t + \varepsilon(uw_x + \Delta V w_Y + ww_z)\} = -p_z,$$
$$u_x + \Delta V_Y + w_z = 0,$$

with $p = \eta$ and $w = \eta_t + \varepsilon(u\eta_x + \Delta V\eta_Y)$ on $z = 1+\varepsilon\eta$, and $w = 0$ on $z = 0$. Here, $\Delta = \varepsilon$ or $\varepsilon^2$, depending on whether (56) or (57) has been used, respectively. (Of course, we could regard $\Delta$ as a third, independent parameter, and this constitutes an alternative route.)

We follow the procedure described in § 6; thus we introduce $\zeta = \sqrt{\varepsilon}(x-t)$, $T = \varepsilon^{3/2}t$, $w = \sqrt{\varepsilon}W$ and redefine $y = Y/\sqrt{\varepsilon\Delta}$, and then expand. (The wave is therefore propagating predominantly in the $x$-direction, with a suitable weak dependence on $y$.) In the case $\Delta = \varepsilon^2$, we obtain

$$2\eta_T + 3\eta\eta_\zeta + \tfrac{1}{3}\delta^2\eta_{\zeta\zeta\zeta} - \tfrac{3}{4}\varepsilon\eta^2\eta_\zeta + \varepsilon V_Y = -\tfrac{1}{12}\varepsilon\delta^2(23\eta_\zeta\eta_{\zeta\zeta} + 10\eta\eta_{\zeta\zeta\zeta}) + O(\varepsilon^2, \delta^4),$$

where

$$V_\zeta = \eta_Y + O(\varepsilon).$$

The corresponding equation for $\hat{u}$, on $z = z_0 = 1/\sqrt{2}$, obtained by introducing $\hat{u}_{xxt}$ and taking $\mu = \frac{5}{6}$ (see the derivation of equation (47)) is

$$\hat{u}_\Theta + \kappa\hat{u}_X + 3\varepsilon\hat{u}\hat{u}_X - \varepsilon\delta^2\hat{u}_{XX\Theta} + \varepsilon^2 V_Y = \varepsilon^2\delta^2(2\hat{u}_X\hat{u}_{XX} + \hat{u}\hat{u}_{XXX}) + O(\varepsilon^3, \varepsilon\delta^4) \tag{58}$$

with

$$V_X = \hat{u}_Y + O(\varepsilon). \tag{59}$$

(We have used the additional scaling transformation $Y \to \frac{1}{2}(\frac{3}{5})^{1/4}Y$.) The conventional structure of the KP equation is evident here, the more so if we eliminate $V$:

$$(\hat{u}_\Theta + \kappa\hat{u}_X + 3\varepsilon\hat{u}\hat{u}_X - \varepsilon\delta^2\hat{u}_{XX\Theta})_X + \varepsilon^2\hat{u}_{YY} = \varepsilon^2\delta^2(2\hat{u}_X\hat{u}_{XX} + \hat{u}\hat{u}_{XXX})_X + O(\varepsilon^3, \varepsilon\delta^4). \tag{60}$$

It should be noted, however, that the $y$-dependence appears here via a term $O(\varepsilon^2)$; in the usual derivation of the KP equation, this term would be $O(\varepsilon)$ in the context of equation (60). This corresponds to the case $\Delta = \varepsilon$ and then, to be consistent with equation (60), higher-order terms in $y$ will have to be included; this particular aspect is not developed here, but it is clear that the resulting equation will be considerably more involved than (60), producing another variant of a two-dimensional CH equation. (Other routes which lead to different higher-dimensional CH equations are described in Holm, Marsden & Ratiu 1998; Kraenkel & Zenchuk 1999; Kraenkel, Senthilvelan & Zenchuck 2000.)

The inclusion of the term in $y$ allows the waves to interact at oblique angles, although the scalings that we have employed imply that, in original variables, the waves are very nearly parallel. For a solitary-wave solution, the effect of the term $\varepsilon^2\hat{u}_{YY}$

is not very dramatic: set $\hat{u} = f(X + lY, \Theta)$, integrate and impose decay conditions, which produces a result equivalent to starting with our CH equation, (49), but with $2\kappa$ replaced by $2\kappa + \varepsilon^2 l^2$. The solitary wave, (54), is then a solution, in the form

$$\hat{u} \sim a\,\mathrm{sech}^2\{\beta(X + lY - c\Theta)\},$$

with $\beta$ and $c$ as before, and $\kappa$ appropriately redefined. The possibility of our two-dimensional CH equation being completely integrable is worth investigating, but well beyond the aims of this study.

## 9. Discussion

We have presented a description of the current approach to the Camassa–Holm equation, via the classical problem of water-wave propagation. In particular, we have given a derivation of the Green–Naghdi equations, their representation in Lie–Poisson Hamiltonian form and the special choice of submanifold that leads to the CH equation for the horizontal velocity component. The two critical assumptions – restricted $z$-structure in the GN equations and the choice of submanifold – have been discussed; in addition, a KdV-type derivation, correct at $O(\varepsilon)$, has been included for comparison. The challenge was then taken up, to obtain the CH equation, in the context of the classical water-wave problem, by using a consistent and appropriate asymptotic approach.

We have shown that, by retaining both the shallowness parameter ($\delta$) and the amplitude parameter ($\varepsilon$), and assuming a double asymptotic expansion, the way is open to develop a CH equation. The approach that we adopt is to retain terms $O(\varepsilon)$, $O(\delta^2)$ and $O(\varepsilon\delta^2)$ – most significantly not $O(\delta^4)$ – and then examine the horizontal velocity component, $u$. We have demonstrated that $u$, evaluated at $z = z_0 = 1/\sqrt{2}$ (where $u = \hat{u}$), and with the judicious inclusion of the term $\hat{u}_{xxt}$, produces a CH equation for $\hat{u}$. (The corresponding equation for the surface wave, $\eta$, to this same order, is not one of the CH family.) A few comments about the solitary-wave solution (both $\mathrm{sech}^2$ and peakon) have been included, particularly the vanishing of the $\mathrm{sech}^4$ term in the expression for $u$ at the special depth $z_0 = 1/\sqrt{2}$.

The results that we have obtained confirm that the CH equation does have a role in the classical water-wave problem. This is especially exciting as the CH equation, like its fairly close neighbour, the KdV equation, is completely integrable. However, we should not lose sight of the fact that the CH may not be uniformly valid. The appearance of singularities in some solutions of the CH equation have indicated that, in the context of a model for water waves, the validity of this equation is open to question. As we have seen, on sufficiently short length scales (measured by $\delta$), the CH equation might be replaced by the higher-order KdV equation. This observation generally, and more particularly near the singularities themselves, provides the opportunity for a careful analysis in the neighbourhood of the singularity. Such an investigation, within the existing formalism leading to the CH equation, is essentially impossible – the relevant higher-order dispersive contributions, for example, are missing – and certainly a wasteful exercise. An area for future study therefore will be, based on our derivation of the CH equation, the behaviour of the full governing equations in the neighbourhood of any singularities of solutions of the CH equation. (We might anticipate that the final conclusion is that the CH equation becomes more like the higher-order KdV equation, but the precise details are for the future.)

Another aspect, which has been explored to a considerable extent in water-wave theory, is the way in which other physical effects distort the simplest, relevant, completely integrable equation. So we are all familiar with the effects of variable depth, underlying 'shear' flow and different geometries, for example, on the KdV equation. Most of these problems are interesting and important, and many give rise to new, completely integrable equations (e.g. KP, concentric KdV). Our derivation of the CH equation, within the context of water waves, enables us to explore a corresponding raft of problems for this equation. As a simple example, we have presented a two-dimensional Camassa–Holm equation; this equation possesses the same structure as the KP equation. The important properties of this equation have yet to be explored (is it completely integrable?), but this does demonstrate that many avenues are now open. Some of these will be pursued in the next phase of the work.

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